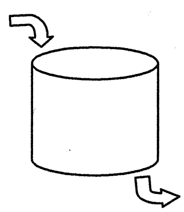
Problem:
A 400 gallon tank initially contains 100 gallons of brine containing 50 pounds of salt. Brine containing 1 pound of salt per gallon enters the tank at a rate of 5 gallons per second, and the well-mixed brine in the tank flows out at the rate of 3 gallons per second. How much salt will the tank contain when it is full of brine?

What we know from the problem:

x(t) = a function which tells us the amount of salt in the tank at time t.

infow information
$r_i = 5$ gal/sec
$c_i = 1$ lb/gal



outflow information
$r_o = 3$ gal/sec
$c_o = ?$

Volume:

Maximum total volume = 400 gal
Initial volume = $V_0 = 100$ gal
Formula: $V(t) = V_0 + (r_i - r_o)t = 100 + 2t$

Concentration (c_o):

Formula: $c_o = \dfrac{x(t)}{V(t)} = \dfrac{x(t)}{100+2t}$

Initial condition: $x(0) = 50$

Solving the Problem for x(t):

For these type of problems we are given the differential equation:

$$\frac{dx}{dt} = r_i c_i - r_o c_o$$

Plugging in our information we get

$$\frac{dx}{dt} = 5 \cdot 1 - 3 \cdot \frac{x}{100+2t}$$

This is a first order linear differential equation, so we put it in the standard form:

$$\frac{dx}{dt} + \left(\frac{3}{100+2t}\right)x = 5, \text{ with } x(0) = 50$$

Notice that $P(t) = \dfrac{3}{100+2t}$ and Q(x) = 5. Thus, we can now solve this ODE

$$\frac{dx}{dt} + \left(\frac{3}{100+2t}\right)x = 5 \text{, with } x(0) = 50$$

Integrating factor: $\rho(t) = e^{\int P(t)dt} = e^{\int\left(\frac{3}{100+2t}\right)dt} = e^{(3/2)\ln(100+2t)} = (100+2t)^{3/2}$

$$(100+2t)^{3/2}\frac{dx}{dt} + (100+2t)^{3/2}\left(\frac{3}{100+2t}\right)x = 5(100+2t)^{3/2}$$

$$\frac{d}{dt}\left(x\cdot(100+2t)^{3/2}\right) = 5(100+2t)^{3/2}$$

$$x\cdot(100+2t)^{3/2} = \int 5(100+2t)^{3/2}\,dt$$

$$x\cdot(100+2t)^{3/2} = (100+2t)^{5/2} + C$$

$$x = 100 + 2t + C(100+2t)^{-3/2}$$

Now, we use our initial condition x(0) = 50

$$50 = x(0) = 100 + C(100)^{-3/2}$$

$$-50 = \frac{C}{1000}, \text{ so } C = -50{,}000$$

Plugging everything back in, we get that

$$x(t) = 100 + 2t - 50{,}000(100+2t)^{-3/2}$$

Answering the question: how much salt does the tank contain when it's full?

Assume that at time T the tank is full of brine.
Then we have that

$$V(T) = 100+2T = 400$$

So, we find that T = 150

By plugging this T into our function x(t), we will find the amount of salt in the tank when it is completely full:

$$x(T) = x(150) = 100 + 2(150) - 50{,}000(100 + 2(150))^{-3/2}$$

$$= 400 - 50{,}000(400)^{-3/2}$$

$$= 400 - 50{,}000(20)^{-3}$$

$$= 400 - \frac{50{,}000}{8000} = 400 - 6.25 = 393.75$$

Thus, we find that x(150) = 393.75
Which means when our tank is full, it contains 393.75 pounds of salt